Recall

- Fourier Inverse formula:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad \text{where} \quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(y) e^{-2\pi i y \xi} dy.$$

- f is of moderate derease if

$$|f(x)| \leq \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

I. Prove Fourier Inverse Formula assuming f is a compactly supported continuous function and $\hat{f}$ is of moderate decrease.

Proof: Assuming $\operatorname{spt} f \in [-M, M]$.

Step 1: For $L > 2M$, for $x \in [-\frac{L}{2}, \frac{L}{2}]$,

$$f(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{L}\right) e^{2\pi i x \frac{n}{L}}$$

Pf: Let h be a L-periodic function s.t. $h(x) = f(x)$ for $x \in [-\frac{L}{2}, \frac{L}{2}]$.

Its Fourier coefficient is given by

$$\hat{h}(n) = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-2\pi i n \frac{x}{L}}\, dx$$

$$= \frac{1}{L}\int_{-\infty}^{\infty} f(x) e^{-2\pi i n \frac{x}{L}}\, dx$$

$$= \frac{1}{L}\hat{f}\left(\frac{n}{L}\right)$$

Recall that if $\sum_{n=-\infty}^{\infty} |\hat{h}(n)| < \infty$, then

$$h(x) = \sum_{n=-\infty}^{\infty} \hat{h}(n) e^{2\pi i x \frac{n}{L}};$$

thus $f(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{L}\right) e^{2\pi i x \frac{n}{L}}$ for $x \in [-\frac{L}{2}, \frac{L}{2}]$.

Note that $\sum_{n=-\infty}^{\infty} |\hat{h}(n)|$

$$= \frac{1}{L}\sum_{n=-\infty}^{\infty} \left|\hat{f}\left(\frac{n}{L}\right)\right|$$

$$\leq \frac{A}{L}\sum_{n=-\infty}^{\infty} \frac{1}{1+\left(\frac{n}{L}\right)^2}$$

$$\leq 2AL\left(\sum_{n=1}^{\infty}\frac{1}{n^2} + 1\right) < \infty$$

Step 2: For any continuous F of moderate decrease,

$$\int_{-\infty}^{\infty} F(x)\,dx = \lim_{\delta \to 0} \sum_{n=-\infty}^{\infty} F(n\delta)\delta$$

Pf: Fix $\varepsilon > 0$. Take $N_1 > 0$ s.t. for $N > N_1$

$$\left| \int_{-\infty}^{\infty} F(x)\,dx - \int_{-N}^{N} F(x)\,dx \right| < \varepsilon$$

By Riemann Integrability,

$$\int_{-N}^{N} f(x)\,dx = \lim_{\delta \to 0} \sum_{|n| \le \frac{N}{\delta}} F(n\delta)\delta$$

However,

$$\left| \sum_{|n| > \frac{N}{\delta}} F(\delta n)\delta \right| \le \sum_{|n| > \frac{N}{\delta}} \delta \, |F(\delta n)|$$

$$\le A \sum_{|n| > \frac{N}{\delta}} \delta \frac{1}{1+\delta^2 n^2}$$

$$\le \frac{A}{\delta} \sum_{|n| > \frac{N}{\delta}} \frac{1}{n^2}$$

$$\le \frac{C}{\delta} \frac{1}{\frac{N}{\delta}} = \frac{C}{N}$$

Take $N_2 > 0$ large s.t. $\frac{C}{N} < \varepsilon$.

Pick $N_0 > \max\{N_1, N_2\}$ and $\delta_0 > 0$ s.t. for any $0 < \delta < \delta_0$,

$$\left| \int_{-N_0}^{N_0} F(x)\,dx - \sum_{|n| \le \frac{N_0}{\delta}} \delta F(\delta n) \right| < \varepsilon.$$

Then

$$\left| \int_{-\infty}^{\infty} F(x)\,dx - \sum_{n=-\infty}^{\infty} \delta F(\delta n) \right|$$

$$\le \left| \int_{-\infty}^{\infty} F(x)\,dx - \int_{-N_0}^{N_0} F(x)\,dx \right|$$

$$+ \left| \int_{-N_0}^{N_0} F(x)\,dx - \sum_{|n| \le} \delta F(\delta n) \right|$$

$$+ \left| \sum_{|n| > \frac{N_0}{\delta}} \delta F(\delta n) \right| < 3\varepsilon.$$

Step 3: Let $F(\xi) = \hat{f}(\xi) e^{2\pi i \xi x}$

Then $\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi$

$$= \int_{-\infty}^{\infty} F(\xi)\, d\xi$$

By step 2 $$= \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{L} f\left(\frac{n}{L}\right) e^{2\pi i x\left(\frac{n}{L}\right)}$$

By step 1 $$= f(x)$$

II. If f, g are of moderate decrease, then $f * g$ is of moderate decrease.

Proof: $$\left| f*g(x) \right| = \left| \int_{-\infty}^{\infty} f(x-y) g(y)\, dy \right|$$

$$= \left| \int_{|y| \le \frac{|x|}{2}} f(x-y) g(y)\, dy + \int_{|y| > \frac{|x|}{2}} f(x-y) g(y)\, dy \right|$$

$$\le \int_{|x-y| \ge \frac{|x|}{2}} |f(x-y)|\,|g(y)|\, dy + \int_{|y| > \frac{|x|}{2}} |f(x-y)|\,|g(y)|\, dy$$

$$\le \frac{A_f \max g}{1 + \frac{x^2}{4}} + \frac{A_g \max f}{1 + \frac{x^2}{4}} \lesssim \frac{1}{1+x^2}$$

$\square$

III. Compute the Fourier transform

(a) $f(x) = e^{-x}\chi_{[0,\infty]}$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

$$= \int_0^{\infty} e^{(-2\pi i \xi - 1)x} dx$$

$$= \frac{1}{-2\pi i \xi - 1}(0 - 1)$$

$$= \frac{1}{2\pi i \xi + 1}$$

(b) $g(x) = e^{-|x|}$

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i \xi x} dx$$

$$= \int_0^{\infty} e^{-x} e^{-2\pi i \xi x} dx + \int_{-\infty}^{0} e^{x} e^{-2\pi i \xi x} dx$$

$$= \frac{1}{1 + 2\pi i \xi} + \frac{1}{1 - 2\pi i \xi}$$

$$= \frac{2}{1 + 4\pi^2 \xi^2}$$

$\square$